OR 644 — Nonlinear Programming
Dr. Hadi El-Amine — Fall 2022
Final exam — Due: 5:00PM on Monday 12/13/2022

1. Answer all questions.
2. Show all your work in order to obtain partial credit.
3. This is an individual assignment. Please work on your own and do not discuss with your classmates.
4. You are not authorized to share the exam material with anyone, including posting it on any website.
5. The points total is 100. **The 5 extra points are bonus**.
6. PLEASE WRITE NEATLY.

1. (20 pts) Use the optimality conditions to find all the local solutions to

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 \\ \text{subject to} \quad & (x_1 - 1)^2 + x_2^2 \le 2 \\ & (x_1 + 1)^2 + x_2^2 \ge 2. \end{aligned}$$

2. (20 pts) Solve the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = c^\top x \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i = 0 \\ & \sum_{i=1}^{n} x_i^2 = 1. \end{aligned}$$

3. (20 pts) Use the active-set method using reduced Newton directions to solve the following problem

$$\begin{aligned} \text{minimize} \quad & f(x) = x_1^2 + 2x_2^2 \\ \text{subject to} \quad & x_1 - x_2 \ge 3 \\ & 2x_1 - 3x_2 \ge 6 \end{aligned}$$

starting from the initial guess $x^0 = (8, 2)^\top$. Plot the progress of your algorithm.

4. (20 pts) Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} \quad & x_1 + 2x_2 \ge 3. \end{aligned}$$

The logarithmic barrier method is used to solve the problem.
a. What is the sequence of unconstrained subproblems that are obtained?

b. What is the solution to each subproblem? (The solution should obviously depend on the barrier parameter.)
c. Do the solutions converge to the optimal solution of the original problem?
d. What are the Lagrange multiplier estimates at each iteration? Do they converge to the optimal Lagrange multipliers?
e. Determine the condition number of the Hessian matrix of the logarithmic barrier function at the solution obtained in part **b.** (The condition number should be a function of the barrier parameter.)
f. Show that, as the barrier parameter approaches zero, the Hessian of the barrier becomes increasingly ill-conditioned.
g. Conceptually, explain the impact of highly ill-conditioned Hessian matrices on the performance of Newton's method.

5. (10 pts) Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = x^\top Q x \\ \text{subject to} \quad & x^\top x = 1, \end{aligned}$$

where Q is a symmetric $n \times n$ matrix.
a. Find all stationary points of the problem.
b. Determine which stationary points are global minimizers.
c. How does your answer in part **a.** change if the constraint is replaced with $x^\top A x \leq 1$, where A is positive definite?

6. (15 pts) A hyperrectangle in $\mathbb{R}^n$ is defined as the set

$$\mathcal{R} = \{x \in \mathbb{R}^n : l_i \leq x_i \leq u_i,\ i = 1, \cdots, n\}.$$

When $n = 2$, set $\mathcal{R}$ is a rectangle and, when $n = 3$, the set is a box. Therefore, $\mathcal{R}$ is simply a generalization of rectangles and boxes to higher dimensional spaces. Consider a polyhedron defined by

$$\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b\}$$

for some $m \times n$ matrix A and a vector $b \in \mathbb{R}^m$. Your goal is to formulate an optimization problem that finds the hyperrectangle $\mathcal{R}$ of largest volume that could be fitted inside $\mathcal{P}$. The decision variables here are obviously vectors l and u in $\mathbb{R}^n$. Your formulation should:

1. be a convex optimization problem. To remind you, a convex optimization problem involves the minimization of a convex function or the maximization of a concave function over a convex feasible region.

2. not contain an exponential number of constraints. In other words, the number of constraints that are included in your problem should not grow exponentially with n. (If you are not able to meet this restriction you will still receive partial credit.)

a. Formulate the problem.
b. Solve and sketch your solution for the following $\mathcal{P}$:

$$\mathcal{P} = \{x \in \mathbb{R}^2 : x_1 + x_2 \geq 0, \quad x_2 \geq 2x_1, \quad x_2 \leq 5\}.$$